Transmission through a smooth parabolic double barrier

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Introduction

The first observation of the resonant tunneling in semiconductor getero-structures [1] induced considerable interest to a wide variety of potentials which can simulate the double-barrier physical structures. For instance, the rectangular [2], triangular [3] and trapezoidal [4] double-barrier potentials were considered. These potentials are not smooth but allow the exact solutions of the Schrödinger equation. The smooth potential was proposed in [5] using Gaussian functions, however this potential does not permit exact analytical solution. The phenomenon of resonant tunneling was also analyzed in the framework of model with the parabolic well between two rectangular barriers [6]. At last, the double-barrier potential was composed with the help of two separated inverted parabolas in [7]. Note that the first derivatives of potentials in [6] and [7] are discontinuous. At the same time the smooth single barrier was constructed in [8] using both parabolas and inverted parabolas. It is not hard to perform transition from the single barrier to the double barrier by means of the simple duplication of the potential profile proposed in [8].

The new symmetric potential function is of the form

$$V(q) = V_0 \begin{cases} 0, & 2q_0 < |q|, \\ \dfrac{(|q| - 2q_0)^2}{(1-g)q_0^2}, & (1+g)q_0 < |q| < 2q_0, \\ 1 - \dfrac{(|q| - q_0)^2}{gq_0^2}, & (1-g)q_0 < |q| < (1+g)q_0, \\ \dfrac{q^2}{(1-g)q_0^2}, & 0 < |q| < (1-g)q_0. \end{cases} \tag{1}$$

Here $0 < g < 1$. The second derivative of the function (1) is discontinuous at the points $q = \mp 2q_0$, $q = \mp(1+g)q_0$ and $q = \mp(1-g)q_0$. However, both the function (1) and its first derivative are continuous. The presence of a varied parameter g allows to change a shape of double-barrier potential in the wide range.

Analytical solution

We are interesting in solving the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + V(q)\right)\Psi(q) = E\Psi(q), \tag{2}$$

where $V(q)$ takes the form (3). It is convenient to introduce dimensionless quantities

$$x = \sqrt{\frac{2mV_0}{\hbar^2}}\, q, \quad x_0 = \sqrt{\frac{2mV_0}{\hbar^2}}\, q_0, \quad e = \frac{E}{V_0}. \tag{3}$$

The transformed Schrödinger equation is given as

$$\left(-\frac{d^2}{dx^2} + v(x)\right)\psi(x) = e\psi(x) \tag{4}$$

with the scaled potential

$$v(x) = \begin{cases} 0, & 2x_0 < |x|, \\ \dfrac{(|x| - 2x_0)^2}{(1-g)x_0^2}, & (1+g)x_0 < |x| < 2x_0, \\ 1 - \dfrac{(|x| - x_0)^2}{gx_0^2}, & (1-g)x_0 < |x| < (1+g)x_0, \\ \dfrac{x^2}{(1-g)x_0^2}, & 0 < |x| < (1-g)x_0. \end{cases} \tag{5}$$

The shape of $v(x)$ is shown in fig. 1 for different values of g when $x_0 = 2$. Here and in subsequent figures we use dotted lines for $g = 0.1$, solid lines for $g = 0.5$ and dashed lines for $g = 0.9$.

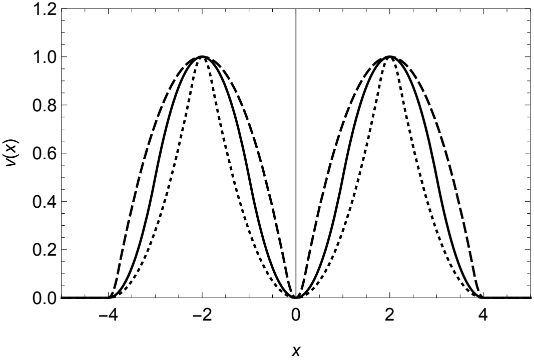

Figure 1: The scaled potential $v(x)$ for different values of g.

The simplicity of the considered potential (5) permits to find the exact solutions of Eq. (4) in five regions. The wave function is represented in the following way

$$\psi(x) = \begin{cases} \exp(i\sqrt{e}\,x) + A_1 \exp(-i\sqrt{e}\,x), & x < -2x_0, \\ A_2\ y_{s1}(z_s^+) + A_3\ y_{s2}(z_s^+), & -2x_0 < x < -(1+g)x_0, \\ A_4\ y_{c1}(z_c^+) + A_5\ y_{c2}(z_c^+), & -(1+g)x_0 < x < -(1-g)x_0, \\ A_6\ y_{s1}(z_s^0) + A_7\ y_{s2}(z_s^0), & -(1-g)x_0 < x < (1-g)x_0, \\ A_8\ y_{c1}(z_c^-) + A_9\ y_{c2}(z_c^-), & (1-g)x_0 < x < (1+g)x_0, \\ A_{10}\ y_{s1}(z_s^-) + A_{11}\ y_{s2}(z_s^-), & (1+g)x_0 < x < 2x_0, \\ A_{12} \exp(i\sqrt{e}x), & 2x_0 < x. \end{cases} \tag{6}$$

There are the incident and reflected waves in the region $x < -2x_0$ and there is the transmitted wave in the region $x > 2x_0$. It is not hard to show that the particular solutions in the region $-2x_0 < x < 2x_0$ are expressed in terms of the confluent hypergeometric functions [9].

In the regions $-2x_0 < x < -(1+g)x_0$, $-(1-g)x_0 < x < (1-g)x_0$ and $(1+g)x_0 < x < 2x_0$, the explicit solutions are given by formulas

$$y_{s1}(z_s) = e^{-z_s^2/4} M\left(\frac{a_s}{2} + \frac{1}{4}, \frac{1}{2}, \frac{z_s^2}{2}\right), \tag{7}$$

$$y_{s2}(z_s) = z_s e^{-z_s^2/4} M\left(\frac{a_s}{2} + \frac{3}{4}, \frac{3}{2}, \frac{z_s^2}{2}\right), \tag{8}$$

$$z_s^{\pm}(x) = \left(\frac{2}{x_0}\right)^{1/2} \frac{(x \pm 2x_0)}{(1-g)^{1/4}}, \quad z_s^0(x) = \left(\frac{2}{x_0}\right)^{1/2} \frac{x}{(1-g)^{1/4}},$$

$$a_s = -\frac{\sqrt{1-g}}{2}\, x_0 e. \tag{9}$$

In the regions $-(1+g)x_0 < x < -(1-g)x_0$ and $(1-g)x_0 < x < (1+g)x_0$, we have the following solutions

$$
\begin{aligned}
y_{c1}(z_c) &= \frac{1}{2}\left\{e^{-iz_c^2/4} M\left(-\frac{ia_c}{2}+\frac{1}{4}, \frac{1}{2}, \frac{iz_c^2}{2}\right)\right. \\
&+ \left. e^{iz_c^2/4} M\left(\frac{ia_c}{2}+\frac{1}{4}, \frac{1}{2}, -\frac{iz_c^2}{2}\right)\right\}, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
y_{c2}(z_c) &= \frac{z_c}{2}\left\{e^{-iz_c^2/4} M\left(-\frac{ia_c}{2}+\frac{3}{4}, \frac{3}{2}, \frac{iz_c^2}{2}\right)\right. \\
&+ \left. e^{iz_c^2/4} M\left(\frac{ia_c}{2}+\frac{3}{4}, \frac{3}{2}, -\frac{iz_c^2}{2}\right)\right\}, \qquad (11)
\end{aligned}
$$

$$
z_c^{\pm}(x) = \left(\frac{2}{x_0}\right)^{1/2} \frac{(x \pm x_0)}{g^{1/4}}, \quad a_c = \frac{\sqrt{g}}{2}\, x_0(1-e). \qquad (12)
$$

It should be stressed that these solutions are real.

By joining the wave function and its first derivative smoothly at six points $x = \mp 2x_0$, $x = \mp(1+g)x_0$ and $x = \mp(1-g)x_0$ we obtain the system of twelve algebraic equations for twelve coefficients A_i. It is easily to solve this system but the solutions are very cumbersome. Therefore we represent only one coefficient

$$A_{12} = \frac{1}{2}\left(\frac{L_- + i\sqrt{e}}{L_- - i\sqrt{e}} - \frac{L_+ + i\sqrt{e}}{L_+ - i\sqrt{e}}\right)\exp\left(-4ix_0\sqrt{e}\right), \qquad (13)$$

where we use notations

$$L_+ = \frac{4}{\sqrt{2x_0}}\frac{1}{(1-g)^{1/4}}\left(\frac{f_{22}}{f_{12}} + \frac{f_{21}}{f_{11}}\right)^{-1},$$

$$L_- = \frac{1}{\sqrt{2x_0}}\frac{1}{(1-g)^{1/4}}\left(\frac{f_{11}}{f_{21}} + \frac{f_{12}}{f_{22}}\right),$$

$$f_{ij} = g^{-1/4}\bar{y}_{si}\bar{y}'_{cj} + (1-g)^{-1/4}\bar{y}_{cj}\bar{y}'_{si}, \quad i = 1,2\,, \quad j = 1,2\,,$$

$$\bar{y}_{si} = y_{si}(\bar{z}_s), \quad \bar{y}'_{si} = \frac{dy_{si}(\bar{z}_s)}{d\bar{z}_s}, \quad \bar{z}_s = \sqrt{2x_0}\,(1-g)^{3/4},$$

$$\bar{y}_{cj} = y_{cj}(\bar{z}_c), \quad \bar{y}'_{cj} = \frac{dy_{cj}(\bar{z}_c)}{d\bar{z}_c}, \quad \bar{z}_c = \sqrt{2x_0}\,g^{3/4}.$$

The square of the absolute value of A_{12} is the transmission coefficient T for the proposed double-barrier potentials (1). The final exact expression is

$$T = |A_{12}|^2 = \left(1 + \frac{d^2}{e}\right)^{-1}, \tag{14}$$

where

$$d = \frac{L_+ L_- + e}{L_+ - L_-}\,. \tag{15}$$

It should be noted that T can be equal to 1 at selected values of e which are the solutions of equation $d(e) = 0$. The resonant tunneling is realized for the incident particle energies less than the barrier height when $e < 1$ ($E < V_0$). Besides, there exist the transmission resonances for energies greater than the barrier height when $e > 1$ ($E > V_0$).

Graphic illustrations

The dependence of the transmission coefficient T on a scaled energy e is given in fig. 2 for $x_0 = 2$ and in fig. 3 for $x_0 = 4$ at different values of g.

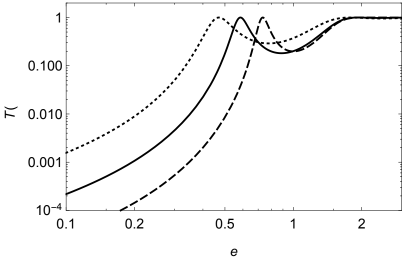

Figure 2: Dependence of T on e for $x_0 = 2$.

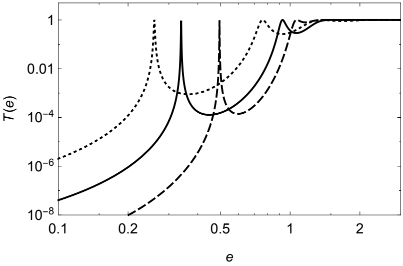

Figure 3: Dependence of T on e for $x_0 = 4$.

The behavior of function $d(e)$ is demonstrated in fig. 4 for $x_0 = 2$ and in fig. 5 for $x_0 = 4$. For example, the resonant tunneling takes place at $e = 0.338815$ and 0.927657 for $0 < e < 1$ and the resonant transmission above barrier takes place at $e = 1.50851$, 2.01586 and 2.40852 for $1 < e < 3$ if $g = 0.5$ and $x_0 = 4$. The transmission coefficient converges to unity at high energies.

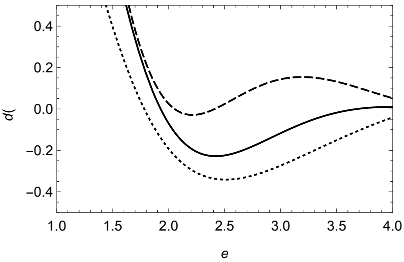

Figure 4: Dependence of d on e for $x_0 = 2$.

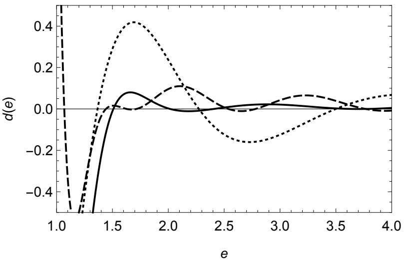

Figure 5: Dependence of d on e for $x_0 = 4$.

At last, the real (solid lines) and the imaginary (dashed lines) components of wave functions are represented in fig. 6 for $x_0 = 2$ and in fig. 7 for $x_0 = 4$ at $e = 0.95$ and $g = 0.5$.

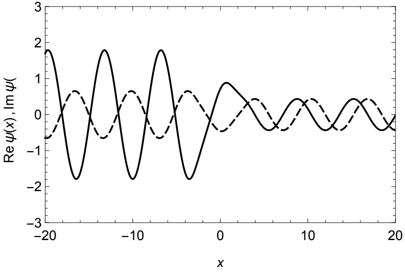

Figure 6: Wave function for $x_0 = 2$.

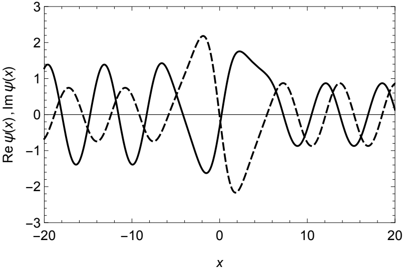

Figure 7: Wave function for $x_0 = 4$.

The proposed parabolic potential extends a short list of exactly solvable models that describe transmission through double barriers. The variable shape of considered barrier gives the wide possibilities to simulate the transmission phenomena. In the present paper, we examined a symmetric potential but it is easily to construct an asymmetric smooth parabolic potential.

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